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Radiation reaction in various dimensions

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Abstract

We discuss the radiation reaction problem for an electric charge moving in flat space-time of arbitrary dimensions. It is shown that four is the unique dimension where a local differential equation exists accounting for the radiation reaction and admitting a consistent mass-renormalization (the Dirac-Lorentz equation). In odd dimensions the Huygens principle does not hold; as a result, the radiation reaction force depends on the whole past history of a charge (radiative tail). We show that the divergence in the tail integral can be removed by the mass renormalization only in the 2+1 theory. In even dimensions higher than four, divergences can not be removed by a renormalization.

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1 Introduction

In this note we address the question whether a generalization of the Dirac-Lorentz equation for a radiating charge in classical electrodynamics exists in space-times of dimension other than four. Apart from a purely academic interest, one is naturally led to higher-dimensional radiation problems in the brane-world set up. Although in the latter case it is not the *vector* field radiation which is interesting to study (vector fields do not live in the bulk), we believe that basic dimensionality conditioned features of radiation reaction are similar for any spin.

If one tries to generalize the Dirac's derivation of the radiation reaction force [1] to arbitrary (flat) space-time dimensions, two kinds of obstacles can be foreseen. First, the divergence of the proper Coulomb potential of a charge in higher dimensions is stronger than in four, so the divergence of the self-force acting on it will be stronger. Another

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unusual feature is encountered in space-times of *odd* dimensions. It is well-known that the Huygens principle does not hold in odd dimensions, and radiation develops a tail, similar to that known in the four-dimensional *curved* space-time [2, 3]. Therefore, it can be expected that the dynamics of a radiating charge in odd dimensions will be governed by an integro-differential equation.

We discuss the difference between Green functions of the scalar D'Alembert equation in neighboring dimensions in detail and suggest a simple physical interpretation of the Huygens principle violation in odd dimensions. The Green function in odd dimensions can be obtained by integrating the neighboring higher even-dimensional Green function over the extra dimension, this is equivalent to take the source smeared along an extra coordinate. Radiation, coming to a given point from distant parts of the line source, produces a tail. Passing to the next lower dimension, one finds that tails are canceled by a destructive interference, when summing up along the second extra dimension.

The case of the radiating charge in $2+1$ (minimal) Maxwell theory is shown to be consistent, and the integral version of the Dirac-Lorentz equation is derived. However all higher-dimensional generalizations, both in even and odd dimensions, fail because of impossibility of absorbing divergences by a renormalization of parameters.

We use the 'mostly minus' metric signature.

2 General setting

For reader's convenience we start by recalling briefly the original Dirac's derivation [1]. Consider the charge equation of motion with a 'bare' mass m_0

$$m_0 \dot{u}^\mu = e F^\mu{}_\nu u^\nu, \tag{2.1}$$

where $u^\mu = dx^\mu$ is a tangent vector to the world-line $x^\mu = x^\mu(s)$, and the field strength denotes the sum $F = F^{ext} + F^{ret}$ of an external field and the (retarded) proper field of the charge. The potential of the latter ($F = dA$) is a solution of the D'Alembert equation

$$\Box A^\mu(x) = -4\pi e \int u^\mu(s) \delta^4(x - x(s)) ds. \tag{2.2}$$

Omitting for the moment the external field, we adopt the standard decomposition of the proper field of the charge $A^{ret} = A^{self} + A^{rad}$:

$$A^{self} = \frac{1}{2}(A^{ret} + A^{adv}), \quad A^{rad} = \frac{1}{2}(A^{ret} - A^{adv}). \tag{2.3}$$

The retarded and advanced Green functions, satisfying the equation

$$\Box G^{ret,adv}(x - x') = -4\pi \delta^4(x - x'), \tag{2.4}$$

in four dimensions are given by

$$G^{ret,adv}(X) = \frac{\delta(T \mp R)}{R}, \tag{2.5}$$

where $X^\mu = x^\mu - x'^\mu$, $T = t - t'$, $R = |\mathbf{r} - \mathbf{r}'|$. Their combinations corresponding to (2.3) are

$$G^{self} = \delta(T^2 - R^2), \quad G^{rad} = \frac{T}{|T|}\delta(T^2 - R^2). \tag{2.6}$$

Taking the value of the electromagnetic field of the charge on its world-line leads to the following integral on the right hand side of (2.1)

$$f^\mu(s) = 4e^2 \int X^{[\mu}(s,s')u^{\nu]}(s')u_\nu(s)\frac{d}{dX^2}G(X)ds', \tag{2.7}$$

where $X^\mu(s,s') = x^\mu(s) - x^\mu(s')$, $X^2 = X^\mu X_\mu = T^2 - R^2$. Due to the presence of delta-functions in both Green functions G^{self} and G^{rad}, only a finite number of Taylor expansion terms in $\sigma = s - s'$ contribute to the integral, since $X^2 = \sigma^2 + O(\sigma^4)$. In the four-dimensional case it is sufficient to retain terms up to σ^3:

$$4X^{[\mu}(s,s')u^{\nu]}(s')u_\nu(s) = \dot{u}^\mu\sigma^2 - \frac{2}{3}(\ddot{u}^\mu + u^\mu\dot{u}^2)\sigma^3 + O(\sigma^4). \tag{2.8}$$

The leading terms in the expansions of derivatives of the Green functions are

$$\frac{d}{dX^2}G^{self}(X) = \frac{d}{d\sigma^2}\delta(\sigma^2), \qquad \frac{d}{dX^2}G^{rad}(X) = \frac{d}{d\sigma^2}\left(\frac{\sigma}{|\sigma|}\delta(\sigma^2)\right). \tag{2.9}$$

Consequently, one encounters the following integrals for the self-force

$$A_l = \int_{-\infty}^{\infty} \sigma^l \, \frac{d}{d\sigma^2}\delta(\sigma^2)\, d\sigma, \tag{2.10}$$

and for the radiation reaction force

$$B_l = \int_{-\infty}^{\infty} \sigma^l \, \frac{d}{d\sigma^2}\left(\frac{\sigma}{|\sigma|}\delta(\sigma^2)\right) d\sigma, \tag{2.11}$$

with $l \geq 2$. The first integral is divergent for $l = 2$, equal to zero for $l = 3$ by parity, and vanishes for all $l > 3$. The second is zero for $l = 2$ by parity and vanishes for all $l > 3$. The integral B_3 is finite, and, in order to disentangle the integrand, it is convenient to regularize the expression by the shift of the argument $\sigma^2 - \epsilon^2$ of the delta-function, taking the limit $\epsilon = 0$ at the end. One finds $B_3 = -1$. The divergent term A_2 enters in the equation of motion multiplied by $\dot{u}^\mu$, so it can be absorbed by the renormalization of the mass,

$$m_0 - A_2 = m, \tag{2.12}$$

and we obtain the Dirac-Lorentz equation

$$m\dot{u}^\mu = eF^\mu{}_\nu u^\nu + \frac{2}{3}(\ddot{u}^\mu + u^\mu\dot{u}^2), \tag{2.13}$$

where the external force term is reintroduced.

This setting remains qualitatively unchanged when we pass to space-times of arbitrary dimensions. But the singularity structure of the Green functions will be different, so in dimensions higher than four we will get larger number of divergent integrals both for the self-force and the radiation reaction force. Moreover, in space-time of odd dimensions the Green functions do not contain delta-functions at all, therefore instead of the differential equation (obtained in the case when only a finite number of powers of σ give a non-zero contribution) we will be left with an integral equation.

3 Green functions in even and odd dimensions

The retarded Green functions of the D'Alembert equation in any dimensions can be presented in a unique way in the momentum representation

$$G_{n+1}^{ret}(t-t',\mathbf{r}-\mathbf{r}') = \frac{2}{(2\pi)^n}\int \frac{\mathrm{e}^{-i\omega(t-t')+i\mathbf{k}(\mathbf{r}-\mathbf{r}')}\, d\omega d\mathbf{k}}{(\omega-i\varepsilon)^2-\mathbf{k}^2}, \tag{3.1}$$

where $\mathbf{r}$, $\mathbf{k}$ are n-dimensional vectors. However, in the coordinate representation, which describes the waves propagation more directly, they turn out to be qualitatively different in odd and even dimensions. Performing a contour integration over ω and then integrating over the $n-1$-dimensional sphere in the $\mathbf{k}$-space one is led to the following representation [4, 5] [1]:

$$G_{n+1}^{ret} = \frac{2\theta(T)}{(2\pi R)^{(n/2-1)}}\int_0^\infty k^{n/2-1}\,\sin kT\; J_{n/2-1}(kR)\,dk. \tag{3.2}$$

Here $T=t-t'$, $R=|\mathbf{r}-\mathbf{r}'|$ and $J_{n/2-1}$ is the Bessel function. Mathematically, the distinction between odd and even dimensions is due to the fact that for odd $n=2\nu+1$ (even dimension of space-time) the index of the Bessel function is semi-integer, and this function is expressible in terms of elementary functions:

$$J_{\nu-1/2}(kR) = \sqrt{\frac{2}{\pi}}(-1)^{\nu-1}\left(\frac{R}{k}\right)^{\nu-1/2}\frac{d^{\nu-1}}{(RdR)^{\nu-1}}\frac{\sin kR}{R}. \tag{3.3}$$

For $\nu=1$ one finds the Green function (2.5) which is localized on the light cone surface. For any other odd $n=2\nu+1$ the result can be obtained by a differentiation

$$G_{n+1}^{ret} = \frac{(-1)^{\nu-1}}{(2\pi)^{\nu-1}}\frac{d^{\nu-1}}{(RdR)^{\nu-1}}\frac{\delta(T-R)}{R}, \tag{3.4}$$

so the localization on the light cone is preserved. In particular, in ten space-time dimensions one has

$$G_{9+1}^{ret} = \frac{1}{(2\pi)^3}\left(\frac{\delta'''(T-R)}{R^4}+\frac{6\delta''(T-R)}{R^5}+\frac{15\delta'(T-R)}{R^6}+\frac{15\delta(T-R)}{R^7}\right). \tag{3.5}$$

[1]The author is grateful to E.Yu. Melkumova for these references.

For even $n = 2\nu$ (odd dimension of space-time) the index of the Bessel function is integer, and an integration in (3.2) no more leads to the delta-function or its derivative. Using the recurrence relation for $J_l(z)$:

$$J_{l+m} = -z^{l+m}\left(\frac{d}{zdz}\right)^m\left(z^{-l}J_l\right), \tag{3.6}$$

one can obtain a relation similar to (3.4) reducing the Green function of $n+1$ theory to that in $2+1$ space-time:

$$G^{ret}_{n+1} \;=\; 2\theta(T)\theta(T^2-R^2)\frac{(-1)^{\nu-1}}{(2\pi)^{\nu-1}}\frac{d^{\nu-1}}{(RdR)^{\nu-1}}\frac{1}{\sqrt{T^2-R^2}}, \tag{3.7}$$

the lowest member of this family being

$$G^{ret}_{2+1} = 2\theta(T)\theta(T^2-R^2)\frac{1}{\sqrt{T^2-R^2}}. \tag{3.8}$$

This function is non-zero *inside* the future light cone. Therefore a signal from a short pulse source attains an observer at a distance R after a time interval R/c, and then follows an infinitely long tail. It is amusing to note how different is the retarded Green function in eleven-dimensional space-time from that in ten dimensions (3.5):

$$G^{ret}_{10+1} \;=\; \frac{210}{(2\pi)^4}\frac{\theta(T^2-R^2)}{(T^2-R^2)^{9/2}}. \tag{3.9}$$

The occurrence of a tail in odd dimensions is by no means surprising. Indeed, let us start with the familiar case of four dimensions representing the retarded Green function in the following form

$$G^{ret}_{3+1} = 2\theta(T)\delta(T^2-R_3^2), \tag{3.10}$$

where for simplicity we have assumed $x'^{\mu} = 0$, so that $R_3^2 = x_1^2+x_2^2+x_3^2$. To pass to the Green function in $2+1$ dimensions it is sufficient to consider, instead of a four-dimensional point-like source $\delta^4(x) = \delta(t)\delta(x_1)\delta(x_2)\delta(x_3)$, a line-like source $\delta^3(x) = \delta(t)\delta(x_1)\delta(x_2)$, that is to integrate the previous source over x_3. Physically, this means that waves coming to the surface of a cylinder $R_2^2 = a^2$, where $R_2^2 = x_1^2+x_2^2$, are collected not only from directions normal to the axis x_3 (these show up exactly after time $t = a$), but also from directions inclined to the axis x_3. These oblique rays propagate longer times $t > a$, up to $t = \infty$, and they constitute a tail. Mathematically, to find a Green function in the next lower dimension one has to integrate the initial Green function over the extra dimension. Integrating the $3+1$-dimensional Green function (2.5) over x_3 one obtains

$$\int G^{ret}_{3+1}dx_3 = \frac{2\theta(T)\theta(T^2-R_2^2)}{\sqrt{T^2-R_2^2}}, \tag{3.11}$$

which is the Green function G^{ret}_{2+1}.

One may wonder why a *double* integration over *two* space-like dimensions leads again to the Green function localized on the light cone and not developing a tail. The answer is simply that tails are canceled by the second integration due to destructive interference. Let us start, e.g., from six dimensions, representing the retarded Green function as

$$G^{ret}_{5+1} = -\frac{2}{\pi}\theta(T)\frac{d}{d\rho^2}\delta(T^2 - R_3^2 - \rho^2), \tag{3.12}$$

where $\rho^2 = x_4^2 + x_5^2$. Now we consider the plane spanned by x_4, x_5 as a source and integrate

$$\int G^{ret}_{5+1} dx_4 dx_5 = \int G^{ret}_{5+1} 2\pi\rho d\rho = 2\theta(T)\delta(T^2 - R_3^2) = G^{ret}_{3+1}. \tag{3.13}$$

The resulting Green function is again localized on the (four-dimensional) light cone. Thus, the line source produces a tail while the plane source does not. This explains 'periodicity' of the Huygens property of the Green functions with varying dimensions.

4 2+1 theory

In odd space-time dimensions, there is no much sense in splitting the retarded Green functions into the self-force part and the radiative part, since contributions from both of them are non-local. Still, the mass renormalization can be consistently performed. Near the coincidence limit the retarded $2+1$ Green function diverges as

$$G^{ret}_{2+1} \sim 2\frac{\theta(\sigma)}{|\sigma|}, \tag{4.1}$$

and therefore for $\sigma \to 0$

$$\frac{d}{dX^2} G^{ret}_{2+1}(X) \sim -\frac{\theta(\sigma)}{|\sigma|^3} \tag{4.2}$$

(the derivatives of the step functions do not contribute to the integral over the world-line). In view of the expansion (2.8), the singularity can be completely removed by the renormalization of the mass term

$$m_0 \dot{u}^\mu + e^2 \int \frac{\dot{u}^\mu(s)}{|s - s'|}\, ds' = m\dot{u}^\mu. \tag{4.3}$$

Now the divergence is logarithmic as could be expected for the two-dimensional Coulomb potential. After this subtraction the remaining integral over the world line becomes finite, and one obtains the following equation for the radiating charge in $2+1$ dimensions

$$m\dot{u}^\mu(s) = eF^\mu{}_\nu u^\nu - e^2 \int_{-\infty}^{s} \left(\frac{2[X^\mu u^\nu(s) u_\nu(s') - u^\mu(s') X^\nu u_\nu(s)]}{(X^2)^{3/2}} - \frac{\dot{u}^\mu(s)}{|s - s'|} \right) \theta(X^2)\, ds'. \tag{4.4}$$

where $X^\mu = X^\mu(s, s') = x^\mu(s) - x^\mu(s')$. The radiation reaction force is given by the integral over an entire history of a charge.

5 $D > 4$

Let us start with even-dimensional space-times. Then the Green functions contain (the derivatives of) $\delta(\sigma^2)$, so only a finite number of Taylor expansions terms in σ will contribute. In dimensions $n+1$ with $n = 2\nu + 1$ we will have instead of (2.9)

$$\frac{d}{dX^2}G^{self}(X) \sim \frac{d^\nu}{(d\sigma^2)^\nu}\delta(\sigma^2), \qquad \frac{d}{dX^2}G^{rad}(X) \sim \frac{d^\nu}{(d\sigma^2)^\nu}\left(\frac{\sigma}{|\sigma|}\delta(\sigma^2)\right). \tag{5.1}$$

Consequently, one encounters the following integrals for the self-force

$$A_l^n = \int \sigma^l \, \frac{d^{(n-1)/2}}{(d\sigma^2)^{(n-1)/2}}\delta(\sigma^2)d\sigma, \tag{5.2}$$

and for the radiation reaction force

$$B_l^n = \int \sigma^l \, \frac{d^{(n-1)/2}}{(d\sigma^2)^{(n-1)/2}}\left(\frac{\sigma}{|\sigma|}\delta(\sigma^2)\right)d\sigma, \tag{5.3}$$

with $l \geq 2$. A_l^n diverges for even l satisfying $2 \leq l < n$, and vanishes for all other l. B_l^n diverges for odd l satisfying $3 \leq l < n$, it has a finite value for $l = n$ and vanishes for all other l.

To calculate the integral over the world-line now one has to expand the quantity (2.8) up to the order n:

$$4X^{[\mu}(s,s')u^{\nu]}(s')u_\nu(s) = \sum_{l=2}^{n} v_l\sigma^l + O(\sigma^{n+1}), \tag{5.4}$$

where v_l are polynomials of derivatives of u^μ of total degree l:

$$v_1 = \dot{u}^\mu, \quad v_2 = -\frac{2}{3}(\ddot{u}^\mu + u^\mu\dot{u}^2), \quad v_3 = \frac{1}{4}\frac{d^3u^\mu}{ds^3} + \frac{1}{6}\dot{u}^\mu\dot{u}^2 + \frac{3}{4}u^\mu(\dot{u}\cdot\ddot{u}), \tag{5.5}$$

etc. To account properly for all powers of σ one also has to expand the argument X^2 of the delta-functions up to higher orders in σ:

$$X^2 = \sigma^2 - \frac{1}{12}\dot{u}^2\sigma^4 + \ldots, \tag{5.6}$$

so that

$$\delta(X^2) = \delta(\sigma^2) - \frac{1}{12}\dot{u}^2\sigma^4\delta'(\sigma^2) + \ldots . \tag{5.7}$$

From this analysis it is clear that, with dimension increasing by two, one acquires two more divergent integrals. Thus, already in six dimensions, we will have three divergent integrals with different functional dependence on dynamical variables, from which only one can be removed by the mass renormalization. It is also worth noting that, if one subtracts the divergent integrals formally, the remaining finite expression for the radiation reaction force would contain n derivatives of the four-velocity. It is unlikely that this quantity could account indeed for the radiation reaction.

Now consider odd-dimensional space-times $n+1$ with $n = 2\nu$. The retarded Green function with both points on the world-line has the leading singularity

$$G^{ret} \sim \frac{\theta(\sigma)}{|\sigma|^{2\nu-1}}. \tag{5.8}$$

To remove all singularities from the integral over the world-line one has to subtract $2\nu-1$ terms of an expansion of the integrand in σ, from which only the lowest subtraction admits an interpretation as a mass-renormalization. The situation is exactly the same, as in even space-time dimensions, although looking somewhat differently.

6 Conclusion

To summarize: we have shown that the four-dimensional space-time is distinguished as the unique one where a local differential equation can be derived for a point charge moving in the external electromagnetic field with account for the radiation reaction. In $2+1$ dimensions the self-energy divergence can be removed by the mass renormalization, but the resulting equation is integro-differential, because of the tail effect in odd-dimensional space-times. In dimensions higher than four, divergences can no longer be removed by the renormalization of parameters, so no consistent generalization of the Lorentz-Dirac equation exists for a point charge. The analysis can be easily extended to curved space-times and radiation fields of other spins.

Clearly, the situation is not better in quantum electrodynamics: this theory is non-renormalizable in dimensions higher than four. But we have shown that even *classical* theory of radiating point charges in higher dimensions is not a fully consistent theory. It would be interesting to investigate classical renormalizability of the equations of motion of p-branes interacting with fields of $p+1$ forms in higher-dimensional space-times.

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